

The Campanelli-Lousto and veiled spacetimes

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The Campanelli-Lousto solutions of Brans-Dicke theory, usually reported as black holes are reconsidered and shown to describe, according to the values of a parameter, wormholes or naked singularities. The veiled Schwarzschild metric recently used as an example to discuss conformal frames and their equivalence corresponds to a special case of the CL metric. The conformal cousins of these solutions, and of the Riegert black hole solution of conformally invariant Weyl theory, are analysed, leading to a word of caution when interpreting physically spacetimes generated via conformal transformations from known seed solutions.

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I. INTRODUCTION

There are relatively few exact solutions of alternative theories of gravity, although many such theories are currently studied with various motivations ranging from the possibility of using them to explain the current acceleration of the universe without dark energy, or for their properties in the early universe, as low-energy effective theories for quantum gravity, as emergent gravity theories, or just as toy models to understand which properties are, or are not, desirable in a theory of gravity (for recent reviews see [1–3]).

When getting to know a theory of gravity, it is important to understand its spherically symmetric solutions and especially its black holes. The prototype alternative to Einstein's theory of General Relativity (GR) was Brans-Dicke theory [4], later generalized to scalar-tensor gravity [5]. An early classification of spherical solutions of Brans-Dicke theory was given by Brans [6]. A common tool used in scalar-tensor and other theories of gravity is that of conformal transformations, which relate the physics in one conformal representation of the theory (“Jordan frame”) to another (“Einstein frame”). Conformal transformations are useful to generate solutions of a theory from known seed solutions of another, but also to relate different solutions within the same theory. There has been a debate about the physical equivalence of conformal frames (see, *e.g.*, [7, 8] and the references therein) and recently the “veiled Schwarzschild spacetime” has been used as an example for discussions of conformal frames [9–11]. Among the known spherically symmetric and static solutions of Brans-Dicke theory are the Campanelli-Lousto spacetimes [12], which are usually reported as black holes or “cold black holes” (be-

cause they have zero surface gravity and temperature [13]). It turns out that the veiled Schwarzschild spacetime is a special case of the Campanelli-Lousto metrics corresponding to certain fixed values of the parameters. Given the widespread use of conformal transformations in gravity and in cosmology, one would like to understand better the veiled Schwarzschild metric and the more general Campanelli-Lousto class to which it belongs, as well as veiled black holes in other theories of gravity. Extra motivation is provided by the finding that generalized Brans-Dicke solutions describe asymptotically Lifschitz black holes in the Jordan (but not in the Einstein) frame [14].

In this paper we will first show (in Sec. II) that the three-parameter Campanelli-Lousto class of solutions of Brans-Dicke theory does not describe black holes. It corresponds, instead, to wormholes or naked singularities, respectively, according to the value of one of the parameters. We then identify, in Sec. III, the veiled Schwarzschild spacetime with a special case of the Campanelli-Lousto class and discuss its properties. A similar analysis is performed in Sec. IV for the veiled Riegert black hole, which is a solution of Weyl's theory of gravity. It is found that caution is needed not to confuse Einstein frame metrics with scaling units of mass, length, and time with their versions with fixed units, and that special care must be taken when interpreting physically even straightforward mathematical results (Sec. V). Sec. VI contains the conclusions.

II. THE CAMPANELLI-LOUSTO SOLUTIONS OF BRANS-DICKE THEORY

To begin with, we recall the Brans-Dicke field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\omega}{\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right)$$

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$$+\frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\square\phi\right), \quad (1)$$

$$\square\phi=0. \quad (2)$$

The Campanelli-Lousto class of spherically symmetric solutions of Brans-Dicke theory [12] is given by¹

$$ds^2 = -V^{b+1}(r)dt^2 + \frac{dr^2}{V^{a+1}(r)} + \frac{r^2}{V^a(r)}d\Omega_{(2)}^2 \quad (3)$$

where

$$V(r) = 1 - \frac{2\mu}{r}, \quad (4)$$

$$\phi(r) = \phi_0 V^{\frac{a-b}{2}}(r), \quad (5)$$

$r > 2\mu$, $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere, $\mu > 0$, a , and b are parameters, and ϕ_0 is a positive constant. We use units in which the speed of light c and Newton's constant G are set equal to unity and we follow the conventions of Ref. [15].

The Brans-Dicke coupling parameter is

$$\omega(a, b) = -2 \frac{(a^2 + b^2 - ab + a + b)}{(a - b)^2}. \quad (6)$$

Taking the trace of (1) and making use of (2), one has

$$R^\mu{}_\mu = \frac{\omega}{\phi^2} \nabla^\alpha \phi \nabla_\alpha \phi. \quad (7)$$

Thus, on the Campanelli-Lousto solution, the Ricci scalar is

$$R^\alpha{}_\alpha = -2 \frac{V^{a-1}(r)}{r^4} \mu^2 (a^2 + b^2 - ab + a + b). \quad (8)$$

The Campanelli-Lousto solutions were believed to be black holes and were presented as such in [12]. However, they correspond to wormholes in a certain region of the parameter space and to naked singularity solutions in other regions. The Campanelli-Lousto class of solutions was rediscovered, with a different parametrization, by Agnese and La Camera [16] who were apparently unaware of [12] and interpreted correctly these solutions as wormholes in the relevant range of parameters. In the following we provide an alternative analysis based on the introduction of the areal radius and the related apparent or trapping horizon.

As a first step, we note that the areal radius² reads

$$R(r) = \frac{r}{V^{a/2}(r)}. \quad (9)$$

We must now distinguish two cases, corresponding to the sign of the parameter a .

A. The case $a \geq 0$

Let us study how the area of 2-spheres of symmetry behaves as r varies. We have

$$\frac{dR}{dr} = \frac{1}{V^{\frac{a+2}{2}}(r)} \left[1 - \frac{(2+a)\mu}{r} \right] \quad (10)$$

and the areal radius (and consequently the area $4\pi R^2$ of 2-spheres which are orbits of the isometry) decreases for $2\mu < r < r_{min} \equiv (2+a)\mu$, has a minimum

$$R_{min} \equiv R(r_{min}) = (2+a)a^{-a/2}\mu \quad (11)$$

at r_{min} , and increases for $r > r_{min}$ (fig. 1). By using the relation between differentials

$$dr = \frac{V^{\frac{a+2}{2}}(r)}{1 - (2+a)\mu/r} dR, \quad (12)$$

it is straightforward to rewrite the Campanelli-Lousto line element (3) as

$$ds^2 = -V^{b+1}(r)dt^2 + \frac{V(r)}{\left[1 - \frac{(2+a)\mu}{r}\right]^2} dR^2 + R^2 d\Omega_{(2)}^2. \quad (13)$$

The apparent (or trapping) horizons of a spherically symmetric spacetime are located by the equation $\nabla^c R \nabla_c R = 0$, corresponding to $g^{RR} = 0$ and $r = r_{min}$. Because of the exponent 2 in the denominator of the coefficient of dR^2 , however, g^{RR} does not change sign at its zero $r = r_{min}$ but has a double root there (see fig. 2). Consider a bundle of radial outgoing null rays which start at $r < r_{min}$ and propagate outward. Their expansion θ_l is positive for $r < r_{min}$ (corresponding to the cross-sectional area of the bundle increasing as one moves along the bundle), it vanishes at $r = r_{min}$ (corresponding to a stationary cross-sectional area of the bundle), and then it is positive again for $r > r_{min}$ (where this area begins increasing again). Therefore, the apparent horizon at r_{min} is not a black hole, but rather a wormhole apparent horizon. Thus, we answer the question posed in Ref. [17] asking whether wormholes supported solely by the Brans-Dicke scalar field are possible when the condition $\omega < -3/2$ is violated.

B. The case $a < 0$

If instead $a < 0$, the areal radius becomes

$$R(r) = rV^{\frac{|a|}{2}}(r) \quad (14)$$

and has a different shape. The minimum of the function (14) occurs at $r_{min} = (2+a)\mu < 2\mu$ and $R(r)$ is always increasing in the relevant range $r > 2\mu$, with $r \rightarrow +\infty$ corresponding to $R \rightarrow +\infty$ (see fig. 3). In this case there is no apparent horizon.

¹ The notation differs slightly from that of Campanelli and Lousto [12]: they denote a with $-n$, b with m , and 2μ with r_0 .

² Note that the areal radius depends on the parameters a and μ , but is independent of b .

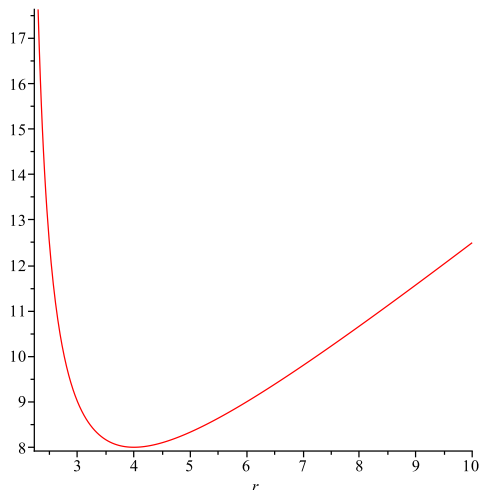


FIG. 1: the areal radius $R(r)$ as a function of r (in units of μ , for the parameter values $\mu = 1$ and $a = 2$).

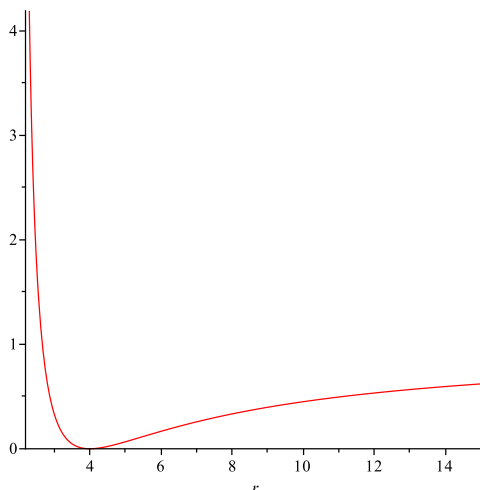


FIG. 2: the metric coefficient g^{RR} as a function of r/μ (for the parameter values $\mu = 1$ and $a = 2$).

The Ricci scalar is given by (8) and, if $a < 1$, diverges as $V \rightarrow 0$ when $r \rightarrow 2\mu^+$. The scalar field (5) also vanishes there. For $a < 1$, therefore, the Campanelli-Lousto spacetimes contain a naked singularity at $R = 0$ (*i.e.*, one that is not enclosed by a black hole horizon). Black holes, wormholes, and naked singularities can in principle be distinguished through the observation of their gravitational lensing effects [18].

C. The case $a = b$

There remains to consider the special case $a = b$. The limit $b \rightarrow a$ corresponds to $\omega \rightarrow \infty$ in eq. (6) and to the scalar field (5) becoming a constant: this is the limit to GR. Formally, in this limit the line element (3) reduces

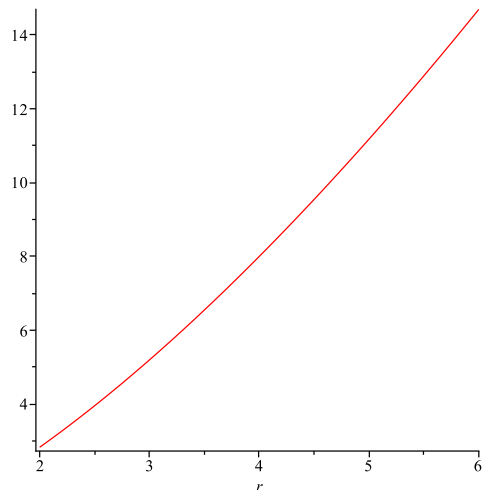


FIG. 3: the areal radius R as a function of r (in units of μ , for the parameter values $\mu = 1$ and $a = -3$).

to

$$ds^2 = -V^{a+1}(r)dt^2 + \frac{dr^2}{V^{a+1}(r)} + \frac{r^2}{V^a(r)} d\Omega_{(2)}^2, \quad (15)$$

which is recognized as the Fisher-Janis-Newman-Winicour line element [19]

$$ds^2 = -V^\nu(r)dt^2 + V^{-\nu}(r)dr^2 + r^2 V^{1-\nu}(r) d\Omega_{(2)}^2 \quad (16)$$

for $\nu = a + 1$. This solution, rediscovered by various authors [20], is the most general static and spherically symmetric solution of the Einstein equations with zero cosmological constant and a massless scalar field [21]. However, by reducing to a constant, the scalar field of the Campanelli-Lousto solution effectively disappears from the stress-energy tensor of the Brans-Dicke massless scalar field, which contains only first and second derivatives of ϕ . Recall that the Brans-Dicke field equations (1) and (2) reduce to the vacuum Einstein equations $R_{\mu\nu} = 0$. Setting the Ricci scalar (8) equal to zero when $b = a$ gives $a = -2, 0$. The value $a = b = 0$ yields immediately the Schwarzschild solution with mass μ . For $a = b = -2$, as noted already in [12], the coordinate change $R = r - 2\mu$ turns again the line element into the Schwarzschild one with mass parameter $m = -|\mu|$. It had to be so because, *in vacuo*, the Jebsen-Birkhoff theorem requires that the only static spherically symmetric solution of $R_{\mu\nu} = 0$ with zero cosmological constant be the Schwarzschild solution. This is also in agreement with a weak version of the Jebsen-Birkhoff theorem in scalar-tensor gravity [22].

III. THE VEILED SCHWARZSCHILD BLACK HOLE

In a recent paper, Deruelle and Sasaki discuss conformal transformations between different conformal frames,

having in mind the Jordan frame and the Einstein frame used in scalar-tensor and $f(R)$ gravity, and argue the physical equivalence of these conformal frames. This paper should be placed in the context of an ongoing debate on the physical equivalence of conformal frames (especially in scalar-tensor gravity (see, *e.g.*, [7, 8] and the references therein and in [9]). Deruelle and Sasaki use as an example the “veiled” Schwarzschild black hole, *i.e.*, they take the Schwarzschild line element of GR

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_{(2)}^2 \quad (17)$$

as the Jordan frame metric and perform a conformal transformation to the Einstein frame metric $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with conformal factor $\Omega = 1/\sqrt{1 - 2M/r}$. The result is the “veiled” Schwarzschild metric

$$d\tilde{s}^2 = -dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + \frac{r^2}{1 - \frac{2M}{r}} d\Omega_{(2)}^2. \quad (18)$$

The veiled Schwarzschild example is used also in [10] to show that the location of an apparent horizon is not invariant under conformal transformations, and in [11] as an explicit example of an apparent (but not event) horizon to question standard beliefs about the thermodynamics of dynamical apparent horizons.

While, in general, conformally transforming a solution of a certain theory of gravity (including GR) does not produce another solution of that theory corresponding to the same form of matter (or to any reasonable mass-energy distribution), the point of [9] is that the conformal transformation generates an equivalent representation of the same physics provided that the scaling of units in the Einstein frame length $\sim \Omega$, time $\sim \Omega$, and mass $\sim \Omega^{-1}$ is taken into account, as explained long ago by Dicke [23]. In other words, when developing a theory of gravity, it must be said how ordinary matter couples to gravity. In the Jordan frame matter is minimally coupled to the spacetime metric, the Einstein equivalence principle holds, and units are fixed. In the Einstein frame, in which the gravitational part of the action assumes the Einstein-Hilbert form, matter is coupled to both the metric and the conformal factor Ω (or the scalar field ϕ) and the units are no longer fixed (this property is true not only in the Einstein frame, but in all the “veiled” frames conformally related to the Jordan frame). The familiar results which follow from the Einstein equivalence principle do not hold in the “veiled” frames and either the notion of varying units or that of variable masses and couplings has to be introduced, expressing the fact that a minimally coupled theory has become a non-minimally coupled one. Then, a conformal transformation does not change the physics. However, one must be careful in using the Einstein frame description since misunderstandings can occur, as explained below.

The veiled Schwarzschild metric is nothing but a special case of the Campanelli-Lousto solutions (3) of Brans-Dicke theory corresponding to the parameter values $a =$

$1, b = -1$, and $\mu = M$. According to the discussion of the previous section, this identification gives the areal radius

$$R(r) = \frac{r}{\sqrt{1 - \frac{2M}{r}}}. \quad (19)$$

This function decreases between $2M < r < r_{min} = 3M$, assumes the minimum value $R_{min} = R(3M) = 3\sqrt{3}M$, and then increases for $r > 3M$. As explained above for the general Campanelli-Lousto class of solutions, the metric (18), taken at face value, describes a wormhole, not a black hole. However, [10] also notes that the quantity which is important to locate when considering conformal transformations is not the apparent horizon (which, contrary to event horizons, changes location under a change of conformal frame). Instead, it is a new surface, characterized in [10] in terms of an entropy 2-form, which is to be considered in place of the apparent horizon when conformal transformations are involved. Indeed, it is noted in [10] that there are no true trapping horizons (at $r = 3M$ or elsewhere) in the veiled Schwarzschild metric (18) and that the expansions of both ingoing and outgoing radial null geodesic congruences vanish at $r = 3M$. When the correct horizon defined through an entropy 2-form is taken into account, the metric (18) is interpreted again as a black hole, not as a wormhole [10]. This fact becomes more intuitive if one thinks that the apparent horizon is the place where the cross-sectional area \mathcal{A} of a bundle of null rays becomes stationary and that, when the units of length are scaling in the Einstein frame, the units of area scale as Ω^2 and one must consider not \mathcal{A} but the ratio of this quantity to its unit, $\sim \mathcal{A}/\Omega^2$. The ratio of the Einstein frame areal radius to its unit is $\sim R/\Omega = r$, which is trivially monotonic and does not describe a wormhole throat. The area of the black hole horizon expressed in varying units of area is

$$\frac{\mathcal{A}}{\Omega^2} = 4\pi R_H^2 \left(1 - \frac{2M}{r_H}\right) = 4\pi r_H^2 = 16\pi M^2, \quad (20)$$

where $r_H = 2M$.

What should we make of all this? If one forgets that the metric (18) is obtained from the Schwarzschild one by means of a conformal transformation, one will naturally consider the apparent horizons of this spacetime instead of the new surface introduced in [10] to characterize its nature. That is, one will correctly interpret the metric (18) as a Campanelli-Lousto solution of Brans-Dicke theory representing a wormhole. However, if the metric (18) is instead seen as a conformally transformed Schwarzschild metric, which represents a genuine black hole (the prototypical one!) and one is aware that the apparent horizon of this metric is not the relevant quantity, but the relevant substitute of this concept is to be located using the prescription in [10] which is conformally invariant, then (18) takes on a new meaning and is interpreted as a black hole instead of a wormhole. Note that both interpretations are correct *given the context to which they refer*, and that this context is very different in

the two situations. If one does not know that (18) comes from the conformal transformation of the Schwarzschild black hole outer region, the only admissible interpretation of (18) is a wormhole spacetime. *Vice-versa*, if it is required that (18) is an Einstein frame metric arising from the conformal transformation of the Schwarzschild black hole, the interpretation is quite different: units are scaling in this frame, and the correct quantity to consider is not the surface where the expansions of null radial geodesic congruences vanish, but the surface introduced in [10].

To summarize, assigning a metric does not tell the whole story about a spacetime: if the metric arises from a conformal transformation of another metric, solution of a certain theory of gravity, this piece of information should be specified as an essential ingredient of the model because it determines the choice of quantities to be studied (*e.g.*, here, the apparent horizon versus the redefined horizon of [10]), the whole context in which to study the metric and, consequently, the physical interpretation of that spacetime metric. This amounts to specifying that ordinary matter is non-minimally coupled to gravity in the Einstein frame.

One way to determine the mass of a black hole or spherical body operationally is by studying the deflection of light in the corresponding spacetime, a technique widely used to map the distribution of dark matter in galaxies and clusters. The famous Einstein formula for the deflection angle of a light ray grazing the surface of a spherical body of mass M and radius R is $\delta\alpha = 4M/R$ and is obtained in the weak-field approximation of the Schwarzschild solution. This deflection angle should also be obtained in the veiled Schwarzschild spacetime. *A priori* this is obvious because null geodesics are conformally invariant but, without this piece of information, it is not immediately clear that this is the case just by looking at the metric (18). Appendix A shows how the calculation of the deflection angle and the empirical determination of the lens mass give the same result in the Schwarzschild and the veiled Schwarzschild spacetimes.

IV. VEILED CONFORMAL GRAVITY BLACK HOLES

The notion of veiled black hole may be generalized and, in order to have again a solution of the system one is dealing with, we consider a conformally invariant theory of gravity. Thus, first we revisit Weyl gravity, namely a conformally invariant quadratic gravity theory and its black hole solution found by Riegert and discussed by others [24, 25]. More precisely, we shall discuss the related topological black hole solutions [26, 27]. To begin with, we write down the action of the model, namely

$$I = \int_{\mathcal{M}} d^4x C^2, \quad (21)$$

where $C^2 \equiv C_{\mu\nu\rho\delta}C^{\mu\nu\rho\delta}$ is the square of the Weyl tensor. This conformally invariant gravity model is very interesting and its phenomenology has been investigated in [28] as a gravitational alternative to dark matter.

We are interested in spherically symmetric static solutions with a topological horizon, namely

$$ds^2 = -W(r)dt^2 + \frac{dr^2}{W(r)} + r^2 d\Sigma_k^2, \quad (22)$$

where

$$d\Sigma_k^2 = \frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\phi^2, \quad (23)$$

and the real parameter k can be $k = 1$ (then the horizon manifold is the usual sphere S_2), $k = 0$ (then the horizon manifold is the torus T_2), or $k = -1$ (in which case the horizon manifold is a compact hyperbolic manifold Y_2).

The topological Riegert solution can be written in the form [27]

$$W(r) = k + 3c_0 - \frac{c_0}{C}(2k + 3c_0)r + br^2 - \frac{C}{r}, \quad (24)$$

where c_0 , C , and b are integration constants. The event horizon exists as soon as there is a positive real solution r_H of the equation $W(r) = 0$. For example, if $C > 0$ and $b = \frac{1}{\ell^2} > 0$, there exists always a positive root. Another simple case is $b > 0$ and $c_0 = 0$, or $b = 0$ and $k = 0$.

As already explained, the related topological “veiled black hole” may be obtained by a conformal transformation with $\Omega = 1/\sqrt{W(r)}$, namely

$$d\tilde{s}^2 = -dt^2 + \frac{dr^2}{W^2(r)} + \frac{r^2}{W(r)} d\Sigma_k^2. \quad (25)$$

In order to analyse this metric, which is still a spherically symmetric static solution of Weyl conformal gravity, we recall the general Hayward formalism [29] and note that the spherically symmetric static spacetime can be written as

$$d\tilde{s}^2 = d\gamma^2 + R^2(x)d\Sigma_k^2, \quad (26)$$

where the normal metric reads

$$d\gamma^2 = \gamma_{ij}(x) dx^i dx^j = -dt^2 + \frac{dr^2}{W^2(r)}. \quad (27)$$

Thus, the new areal radius is

$$R(r) = \frac{r}{\sqrt{W(r)}}. \quad (28)$$

The coordinate r has a range such that $W(r) > 0$. As a function of r , $R(r)$ diverges when $r \rightarrow r_H$. The trapping horizon is defined by

$$\gamma^{ij}\partial_i R \partial_j R = W^2(r) (R'(r_0))^2 = 0, \quad (29)$$

and this reduces to the extremal condition $R'(r) = 0$ for $R(r)$, namely

$$W(r_0) = \frac{r_0}{2} \frac{dW_0}{dr}. \quad (30)$$

This condition defines a double “trapping” horizon at $r = r_0$, but without a trapped region since the two expansions of ingoing and outgoing radial null geodesic congruences are both vanishing at $r = r_0$ (see Appendix B for further discussion). Making use of the Hayward invariant surface gravity, which can be defined in the presence of a generic trapping horizon

$$\kappa_H = \frac{1}{2} \square_\gamma R|_H, \quad (31)$$

one has in our static case

$$\kappa_H = \frac{1}{2} W_0^2 R_0''. \quad (32)$$

The second derivative of the areal radius at the apparent horizon is

$$R_0'' = \frac{1}{2W_0^{3/2}} (W_0' - r_0 W_0'') \quad (33)$$

Then, $R_0'' > 0$ if and only if $\kappa_H > 0$. Thus, R has a local minimum and is locally a wormhole since the lapse function is trivially constant. We may write

$$\kappa_H = \frac{1}{4} W_0^{1/2} (W_0' - r_0 W_0''). \quad (34)$$

As an example, for the veiled Schwarzschild black hole, where $W(r) = 1 - \frac{2M}{r}$, one has $\kappa_H = \frac{1}{6\sqrt{3}M}$ in agreement with [11].

A further local characterization may be achieved by introducing the Kodama vector, which is a natural generalization of the Killing vector ∂_t to spherically symmetric static spacetimes. It is defined on the normal space and is trivially extended to the whole spacetime, namely

$$K^i = \frac{\varepsilon^{ij} \partial_j R}{\sqrt{-\gamma}}. \quad (35)$$

In our case, one has

$$K(r) = W(r) R'(r) \partial_t, \quad (36)$$

and at the double horizon $r = r_0$ where $R'_0 = 0$, it is $K_0 = 0$, which is a typical local property of wormholes.

Finally, we can investigate the asymptotic behaviour. When $r \rightarrow r_H$, the radius of the original black hole event horizon, the veiled metric (25) reads

$$d\tilde{s}^2 = -dt^2 + \frac{1}{\kappa_H^2 u^2} (du^2 + r_H^2 d\Sigma_k^2) \quad (37)$$

where the new coordinate u is small, being defined by

$$u^2 = \frac{4}{W_H'} (r - r_H), \quad (38)$$

and $\kappa_H = W_H'/2$ is the surface gravity of the original black hole. For example for $k = 0$, corresponding to an initial toroidal black hole, one has locally $\mathbb{R} \times \mathbb{H}^3$, \mathbb{H}^3 being the hyperbolic 3-dimensional space.

With regard to the other limit, it depends on the range of r . For example, if one assumes that $b > 0$ and the original black hole is asymptotically anti-de Sitter then, when $r \rightarrow r_H$ (the original black hole event horizon radius), the veiled metric reads

$$d\tilde{s}^2 = -dt^2 + \frac{1}{b^2} dv^2 + \frac{d\Sigma_k^2}{b}, \quad (39)$$

with the new coordinate defined by $v = 1/r$, thus large r corresponds to small values of v .

V. CAMPANELLI-LOUSTO SPACETIMES IN GR ARE FISHER-JANIS-NEWMAN-WINICOUR

If the veiled Schwarzschild metric (18) is regarded as the conformal transform of the Schwarzschild black hole in an Einstein frame, with the information that all units are scaling and only ratios of quantities to the respective units are to be considered as physical (but not the quantities themselves), then (18) is interpreted as describing a black hole. If instead the line element (18) is taken without this information, *i.e.*, with fixed units, it is correctly interpreted as describing a wormhole spacetime. This situation, to which we arrived by contemplating the line element (18), has a counterpart in the general case of the Campanelli-Lousto solution (3), of which (18) is a special case. That is, suppose that we perform the standard conformal transformation to the Einstein frame of Brans-Dicke theory $(g_{\mu\nu}, \phi) \rightarrow (\tilde{g}_{\mu\nu}, \tilde{\phi})$ with

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = \phi g_{\mu\nu}, \quad (40)$$

$$\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi}} \ln \phi. \quad (41)$$

Then the vacuum Brans-Dicke action

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) \quad (42)$$

is cast into the Einstein frame form

$$S_{BD} = \int d^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} \right), \quad (43)$$

where a tilde denotes Einstein frame quantities. If one forgets about the origin of this action and the scaling of units in the Einstein frame, one will interpret (43) as the action of GR with a minimally coupled and non self-interacting scalar field $\tilde{\phi}$ and fixed units. Then, the solutions of the vacuum Brans-Dicke theory will correspond to solutions of GR with this scalar field. The Campanelli-Lousto class of solutions (3) generates a corresponding

class of solutions of GR with a free minimally coupled scalar field (and fixed units) which coincides with the Fisher-Janis-Newman-Winicour [19] class, as is shown below. The lesson is that, when the information that varying units should be used in the Einstein frame is dropped, the Campanelli-Lousto wormholes do not correspond to black holes of Einstein theory.

Let us see how this situation occurs. The Campanelli-Lousto line element conformally rescaled according to eq. (40) is

$$d\tilde{s}^2 = -V^{\frac{a+b+2}{2}}(r)dt^2 + \frac{dr^2}{V^{\frac{a+b+2}{2}}(r)} + \frac{r^2}{V^{\frac{a+b}{2}}(r)}d\Omega_{(2)}^2, \quad (44)$$

where eq. (5) has been used and an irrelevant multiplicative constant has been dropped. According to eq. (41), the minimally coupled free scalar field sourcing this metric is

$$\tilde{\phi} = \frac{\pm(a-b)}{8\sqrt{\pi}}\sqrt{-(a+b)(a+b+4)}\ln V(r) \quad (45)$$

if $b \neq a$. In order for this expression to be real, the argument of the square root must be non-negative, which corresponds to $\omega > -3/2$ and yields

$$-4 \leq a+b \leq 0. \quad (46)$$

The solution depends only on the combination $\delta \equiv \frac{a+b}{2}$ and not on a and b separately and it is recognized to be of the Fisher-Janis-Newman-Winicour form (16) with $\nu = \delta$, subject to the constraint $-2 < \nu < 0$. Since the Fisher-Janis-Newman-Winicour metric is the most general static and spherically symmetric solution of the Einstein equations with zero cosmological constant and a massless scalar field as a source [21], the conformal transformation of the Campanelli-Lousto line element cannot give a new solution of GR. The areal radius is

$$R(r) = \frac{r}{V^{\frac{a+b}{4}}} \quad (47)$$

(note that this expression depends on all the three parameters a, b , and μ , contrary to the one in eq. (9)) and, since

$$\frac{dR}{dr} = V^{-\frac{(a+b+4)}{4}}(r) \left[1 - \frac{\mu}{r} \left(\frac{a+b+4}{2} \right) \right], \quad (48)$$

formally the minimum of the function $R(r)$ is at $r_0 = \left(\frac{a+b+4}{2}\right)\mu$. However, the range of the radial coordinate r in the line element (3) is $r > 2\mu$ and $r_0 > 2\mu$ implies $a+b \geq 0$, in contradiction with eq. (46). Therefore, it is always $r_0 < 2\mu$ and the areal radius $R(r) = rV^{\frac{|a+b|}{2}}(r)$ is always an increasing function of r for $r > 2\mu$, with $\lim_{r \rightarrow 2\mu^+} R(r) = 0$.

Now, using the relation between the differentials of the radial coordinates

$$dr = \frac{rV^{\frac{a+b+4}{4}}(r)}{1 - r_0/r} dR, \quad (49)$$

one obtains the line element

$$d\tilde{s}^2 = -V^{\frac{a+b+2}{2}}(r)dt^2 + \frac{V(r)}{(1 - \frac{r_0}{r})^2} dR^2 + R^2 d\Omega_{(2)}^2. \quad (50)$$

Since we are now considering GR with fixed units (which is a very different context from Brans-Dicke theory in the Einstein frame with varying units), the apparent horizons are located by the equation $g^{RR} = 0$ or

$$\frac{(1 - r_0/r)^2}{V(r)} = 0 \quad (51)$$

which has no roots for $r > 2\mu$ since $r_0 < 2\mu$. There are no apparent horizons and the GR-with-fixed-units solutions generated by the Campanelli-Lousto metrics (3) only correspond to the second class of spacetimes discussed in Sec. II B and to a subclass of the Fisher-Janis-Newman-Winicour solutions.

The Ricci scalar is now

$$R^\mu{}_\mu = \frac{\mu^2 V^{\frac{a+b-2}{2}}}{2r^4} (a-b)^2 [-(a+b)(a+b+4)], \quad (52)$$

and if $a+b < 2$, which is the case here, diverges as $V \rightarrow 0$ when $r \rightarrow 2\mu$ and $R \rightarrow 0$. Also the scalar field (45) diverges in this limit and the spacetime described by the line element (50) contains a naked singularity at $R = 0$. It is well-known that the Fisher-Janis-Newman-Winicour solution exhibits a naked singularity at $r = 2\mu$. This result is consistent with that of Ref. [30] whose authors find that adding a scalar field to the exterior Reissner-Nordström or Kerr solutions shrinks the event horizon to a point. Ref. [31] studies a more general metric in the Einstein frame and provides conditions for it to describe a wormhole. The line element (44) is less general than the spherically symmetric metric of [31] and does not admit wormholes. Our discussion of Sec. II is an (expanded) Jordan frame version of the Einstein frame discussion of [31].

VI. CONCLUSIONS

We have studied known solutions of Brans-Dicke theory, Weyl theory, and GR related by conformal mappings. First, the Campanelli-Lousto solutions of Brans-Dicke theory which are believed to describe black holes, are shown instead to correspond to wormhole spacetimes for positive values of the parameter a , and to spacetimes containing a naked singularity at $R = 0$ when $a < 1$. Then, we realized that the veiled Schwarzschild metric used as an example in the discussion of the physical equivalence of conformal frames coincides with the Campanelli-Lousto solution of Brans-Dicke theory (3) for the parameter values $a = 1, b = -1$, and $\mu = M$. When the Campanelli-Lousto metrics are mapped to the Einstein frame and their conformal cousins are regarded as GR solutions (*i.e.*, with fixed units), they always generate a subclass of

the Fisher-Janis-Newman-Winicour solution containing a naked singularity. The lack of a one-to-one correspondence between black holes in the Jordan and Einstein frames (in the absence of scaling units) was already noted in [13, 32], although the difference between scaling units and fixed units was not noted. The moral is that there is a big difference between the Einstein frame with varying units of time, length and mass (and of course, derived units scaling in the appropriate way) and GR with fixed units. The Campanelli-Lousto and veiled Schwarzschild spacetimes demonstrate this difference. We have then studied the veiled version of the Riegert black hole solution of Weyl gravity, which is a conformally invariant theory. Even in this situation, the conformal transformation of the Riegert black hole generates, as a solution of the same theory, a wormhole without trapped regions.

Our discussion induces a word of caution when using conformal transformations to an Einstein frame and drawing physical interpretations of the mathematical results. In such situations, properties which seem obvious are not always true.

Appendix A: Empirical determination of the mass of unveiled and veiled black holes using light deflection

Consider a photon starting at spatial infinity which, in the absence of a gravitational lens, would have as unperturbed path the z -axis and the unperturbed four-momentum

$$p_{(0)}^\mu = (1, 0, 0, 1) = \delta_0^\mu + \delta_3^\mu$$

in Cartesian coordinates (t, x, y, z) . Let us introduce now a spherical lens of mass M described by the Schwarzschild metric (17) and assume that lensing occurs in the weak-field regime. The photon four-momentum is now $p^\mu = p_{(0)}^\mu + \delta p^\mu$ and satisfies the null geodesic equation

$$\frac{dp^\mu}{d\lambda} + \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta = 0$$

which, to first order in the deflections, reduces to

$$\frac{d(\delta p^\mu)}{d\lambda} + \Gamma_{\alpha\beta}^\mu p_{(0)}^\alpha p_{(0)}^\beta = 0,$$

where

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} \eta^{\mu\sigma} (h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma})$$

and the metric is expressed as the Minkowski metric plus perturbations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Integrating along the photon path between the source and the observer gives

$$\delta p^\mu = - \int_S^O d\lambda \Gamma_{\alpha\beta}^\mu p_{(0)}^\alpha p_{(0)}^\beta \simeq - \int_S^O dz \Gamma_{\alpha\beta}^\mu p_{(0)}^\alpha p_{(0)}^\beta,$$

where to first order it is legitimate to approximate the photon path with the unperturbed path (the z -axis) along which $\lambda \simeq z = t$.

In the veiled Schwarzschild spacetime (18) the null geodesic equation is still satisfied by the four-momentum \tilde{p}^μ of the photon,

$$\frac{d(\delta \tilde{p}^\mu)}{d\lambda} + \tilde{\Gamma}_{\alpha\beta}^\mu \tilde{p}_{(0)}^\alpha \tilde{p}_{(0)}^\beta = 0.$$

Strictly speaking, a conformal transformation can change an affinely-parametrized geodesic into a non-affinely parametrized one, but it is possible to reparametrize the null geodesic using an affine parameter such that the z -axis coincides with the unperturbed photon path and $\tilde{p}_{(0)}^\mu = p_{(0)}^\mu = \delta_0^\mu + \delta_3^\mu$, and we assume that this has been done. Using the relation between the Christoffel symbols of conformally related spacetimes

$$\tilde{\Gamma}_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu + \frac{1}{\Omega} \left(\delta_\alpha^\mu \partial_\beta \Omega + \delta_\beta^\mu \partial_\alpha \Omega - g_{\alpha\beta} \partial^\mu \Omega \right),$$

one obtains

$$\begin{aligned} & \frac{d(\delta \tilde{p}^\mu)}{d\lambda} + \Gamma_{\alpha\beta}^\mu p_{(0)}^\alpha p_{(0)}^\beta \\ & + \left[\delta_\alpha^\mu \partial_\beta (\ln \Omega) + \delta_\beta^\mu \partial_\alpha (\ln \Omega) - g_{\alpha\beta} \partial^\mu (\ln \Omega) \right] p_{(0)}^\alpha p_{(0)}^\beta. \end{aligned}$$

Integrating along the unperturbed photon path between source and observer yields

$$\delta \tilde{p}^\mu = \delta p^\mu$$

$$- \int_S^O dz \left[\delta_\alpha^\mu \partial_\beta (\ln \Omega) + \delta_\beta^\mu \partial_\alpha (\ln \Omega) - g_{\alpha\beta} \partial^\mu (\ln \Omega) \right] p_{(0)}^\alpha p_{(0)}^\beta$$

and using the fact that

$$p_{(0)}^\alpha p_{(0)}^\beta = \delta_0^\alpha \delta_0^\beta + \delta_0^\alpha \delta_3^\beta + \delta_3^\alpha \delta_0^\beta + \delta_3^\alpha \delta_3^\beta,$$

it is

$$\delta \tilde{p}^\mu = \delta p^\mu - 4\delta_0^\mu [\ln \Omega]_S^O - 4\delta_3^\mu [\ln \Omega]_S^O.$$

Since the light source and the observer are in the asymptotic region, the terms in square brackets vanish and one obtains $\delta \tilde{p}^\mu = \delta p^\mu$, as expected from the independent knowledge that null geodesics are conformally invariant. This result then produces the same deflection angle and the same operational determination of lens mass in the Schwarzschild metric and in the veiled Schwarzschild spacetime.

Appendix B: The light sphere of the Reigert black hole

Here we discuss a geometrical interpretation of the double trapping horizon of a veiled black hole-wormhole with regard to the so-called light sphere of the original unveiled black hole.

We recall that, given a spherically symmetric black hole solution in the form

$$ds^2 = -W(r)dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega_2^2,$$

one can determine the associated light sphere by studying the equation of motion of classical relativistic massless particles. For a massless particle, it is well-known that the associated Lagrangian may be written in the form

$$L = \frac{1}{2V} \frac{ds^2}{d\lambda^2},$$

V being the einbein and λ a suitable evolution parameter. We may deparametrize this relativistic reparametrization-invariant system by making the choice $d\lambda = d\phi$, for angular variable. Then, choosing the other angular variable $\theta = \pi/2$, eliminating the einbein V , and making use of two other constants of motion k_0 and h associated with the conservation of energy and angular momentum, respectively, a textbook approach provides the equation of motion for the trajectory as well as the first integral of motion

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 W(r) = \frac{k_0^2}{h^2} r^4.$$

As is well-known, it is convenient to make use of the Newton variable $u \equiv 1/r$. Then, the equation of motion of the light trajectory reduces to

$$\frac{d^2 u}{d\phi^2} + uW(u) + \frac{u^2}{2} \frac{dW}{du} = 0.$$

The light sphere is defined by $u = u_0$, describing a circular trajectory with constant radius. As a result, the radius is determined by

$$u_0 W_0 + \frac{u_0^2}{2} \frac{dW_0}{du} = 0.$$

In terms of the original radial coordinate r , one has

$$W(r_0) = \frac{r_0}{2} \frac{dW_0}{dr},$$

which corresponds to the extremal property of the areal radius of the related “veiled black hole”

$$d\tilde{s}^2 = -dt^2 + \frac{dr^2}{W^2(r)} + \frac{r^2}{W(r)} d\Omega_2^2.$$

This property still holds true for a topological black hole. However, the light sphere may not exist. For example, let us consider the topological black holes in the presence of negative or positive cosmological constant Λ [33–35] where

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Sigma_k^2,$$

with

$$V(r) = k - \frac{C}{r} - \frac{\Lambda}{3} r^2,$$

and where C is a mass parameter, which in the case $k = -1$ may be negative. The radius of the light sphere must satisfy

$$r_0 = \frac{3}{2k} C.$$

Note that this condition does not depend on Λ . For example, for $k = 0$ (toroidal black hole), there is no finite light sphere. For $k = 1$, r_0 must be bigger than the event horizon, and for $\Lambda = 0$, $C = 2M$, one obtains the well-known result for the Schwarzschild solution. For $k = -1$ and $\Lambda < 0$, the condition may be satisfied for a restricted range of the mass parameter C .

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